Exercise set 2 – Introduction to kinematics - Solutions

Reminders

Simplified notation of sines and cosines

To simplify the notation, we use:

- $\sin(\theta) = s$
- $\cos(\theta) = c$
- $\sin(\theta_1) = s_1$
- $\cos(\theta_1) = c_1$
- $\sin(\theta_2) = s_2$
- $\cos(\theta_2) = c_2$
- $\cos(\theta_1 + \theta_2) = c_{1+2}$
- $\sin(\theta_1 + \theta_2) = s_{1+2}$

Rotation and translation matrices

Recall that:

- $\mathbf{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ describes the rotation of θ around the origin (in 2D)
- $\mathbf{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$ describes the translation vector $\mathbf{t}$

Trigonometry of the sum of two angles

Recall as well that:

$$\cos(a + b) = \cos a \,.\, \cos b - \sin a \,.\, \sin b$$
$$\sin(a + b) = \sin a \,.\, \cos b + \cos a \,.\, \sin b$$

Sequence of transformations

Finally, the sequence $a \rightarrow b \rightarrow c$ describes the transformation a followed by the transformation b followed by the transformation c.

Exercise 1: 2D rotations around the origin

Calculate the following 2D rotation matrices:

1. $\mathbf{R}(\theta = 0)$.
2. $\mathbf{R}(-\theta)$.
3. $(\mathbf{R}(\theta))^{-1}$.
4. $\mathbf{A} = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$. Also, find θ such that $\mathbf{A} = \mathbf{R}(\theta)$.
5. Determine if the following equality is true: $\mathbf{R}(\theta_1)\mathbf{R}(\theta_2) = \mathbf{R}(\theta_2)\mathbf{R}(\theta_1)$.

Solution 1

1.
$$\mathbf{R}(\theta = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.
$$\mathbf{R}(-\theta) = \begin{pmatrix} c_{-\theta} & -s_{-\theta} \\ s_{-\theta} & c_{-\theta} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

3. Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, therefore :

$$\begin{aligned} \left(\boldsymbol{R}(\theta)\right)^{-1} &= \frac{1}{c^2 + s^2} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\ &= \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \left(\boldsymbol{R}(\theta)\right)^T \\ &= \boldsymbol{R}(-\theta) \end{aligned}$$

4.
$$\begin{aligned} \mathbf{A} &= \mathbf{R}(\theta_2)\,\mathbf{R}(\theta_1) \\ &= \begin{pmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix} \\ &= \begin{pmatrix} c_2c_1 - s_2s_1 & -c_2s_1 - s_2c_1 \\ c_2s_1 + s_2c_1 & c_2c_1 - s_2s_1 \end{pmatrix} \\ &= \begin{pmatrix} c_{1+2} & -s_{1+2} \\ s_{1+2} & c_{1+2} \end{pmatrix} \\ &= \mathbf{R}(\theta_1 + \theta_2) \end{aligned}$$

5.
$$\begin{aligned} \mathbf{R}(\theta_2)\mathbf{R}(\theta_1) &= \mathbf{R}(\theta_1 + \theta_2) \\ &= \mathbf{R}(\theta_2 + \theta_1) \\ &= \mathbf{R}(\theta_1)\mathbf{R}(\theta_2) \end{aligned}$$

Exercise 2: Homogeneous transformation matrices in 2D

1. Give the homogeneous matrix corresponding to the pure translation $\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$
2. Give the homogeneous matrix corresponding to the pure rotation $\boldsymbol{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$
3. Give the homogeneous matrix for the following sequence:

$$\mathbf{t} \rightarrow \mathbf{R}(\theta)$$

4. Give the homogeneous matrix for the following sequence:

$$\mathbf{R}(\theta) \rightarrow \mathbf{t}$$

5. Does the homogeneous matrix $\begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$ correspond to the sequence of a translation and a rotation, or to the sequence of a rotation and a translation?
6. Give the homogeneous matrix corresponding to the sequence opposite to that of point 4:

$$\mathbf{R}(-\theta) \rightarrow -\mathbf{t}$$

Solution 2

1. Homogeneous matrix of a pure translation :

$$\mathbf{M}_t = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

2. Homogeneous matrix of a pure rotation with an angle θ around the origin :

$$\mathbf{M}_r = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Homogeneous matrix for the sequence $\mathbf{t} \rightarrow \mathbf{R}(\theta)$:

$$\begin{aligned}
\mathbf{M}_r\mathbf{M}_t &= \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c & -s & ct_x - st_y \\ s & c & st_x + ct_y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{R}(\theta)\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}
\end{aligned}$$

4. Homogeneous matrix for the sequence $\mathbf{R}(\theta) \rightarrow \mathbf{t}$:

$$
\begin{aligned}
\mathbf{M}_t\mathbf{M}_r &= \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}
\end{aligned}
$$

5. It corresponds to the sequence of a rotation followed by a translation, i.e the previous point.

6. Homogeneous matrix for the sequence $\mathbf{R}(-\theta) \rightarrow -\mathbf{t}$:

$$
\begin{aligned}
\mathbf{M}_{-t}\mathbf{M}_{-r} &= \begin{pmatrix} \mathbf{R}(-\theta) & -\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{R}\,(\theta))^T & -\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} c & s & -t_x \\ -s & c & -t_y \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Exercise 3: Sequence of homogeneous matrices

Give the homogeneous matrix for the following sequence of operations:

$$\mathbf{R}(\theta_1) \rightarrow \mathbf{t_1} \rightarrow \mathbf{R}(\theta_2) \rightarrow \mathbf{t_2}$$

Solution 3

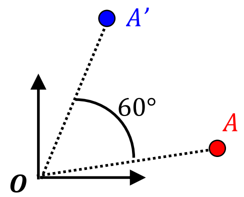
We have :

$$
\begin{aligned}
\mathbf{M} &= \begin{pmatrix} \mathbf{R}_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_2 & -s_2 & t_{x2} \\ s_2 & c_2 & t_{y2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & t_{x1} \\ s_1 & c_1 & t_{y1} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{1+2} & -s_{1+2} & c_2t_{x1} - s_2t_{y1} + t_{x2} \\ s_{1+2} & c_{1+2} & s_2t_{x1} + c_2t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2 \\ \mathbf{0} & 1 \end{pmatrix}
\end{aligned}
$$

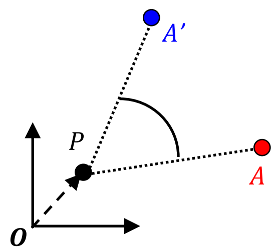
Exercise 4: Rotations around an arbitrary point

Any combination of translations and rotations in the plane can be expressed as a pure rotation around a point P, called the center of rotation.

1. Is the statement above entirely correct? What about a pure translation?
2. For a combination of translations and rotations:
   (a) Show how the pure center of rotation P can be found graphically. Hint: draw a random vector ***v*** in a plane, and also the result of its homogeneous transformation, then find the pure center of rotation P by construction.
   (b) Find the center of rotation analytically, using the formula for a rotation around a point P given by the vector $\mathbf{p}$. Consider the homogeneous matrix of transformation for the combination of translations and rotations as known.
   (c) Find the same center of rotation by reasoning about what happens when the transformation is applied to this center of rotation. Again, consider the homogeneous matrix of transformation for the combination of translations and rotations as known.
3. Give the homogeneous matrix which describes a rotation of $60°$ around the origin.



4. Give the homogeneous matrix which describes a translation of 1 unit in the *x* direction, then a rotation of 60° around the origin.
5. Give the homogeneous matrix which describes a rotation of $60°$ around the point $P = \binom{1}{1}$.



6. An object with two points v_1 and w_1 is moved so that the images of the points are found respectively at positions v_2 and w_2 with respective homogeneous coordinates:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} \frac{2-\sqrt{3}}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Give the homogeneous matrix $\mathbf{M}$ which describes this transformation.

Solution 4

1. Yes, the special case of pure translation can be seen as rotation with a center of rotation at infinity.

   As an example, consider the image P_2 of point $P_1 = \binom{0}{0}$ at the origin, resulting from a rotation of angle θ around a point $C = \binom{R}{0}$:

$$P_2 = \begin{pmatrix} R \cdot (1 - \cos\theta) \\ -R \cdot \sin\theta \end{pmatrix}$$

   We can rewrite the equation as :

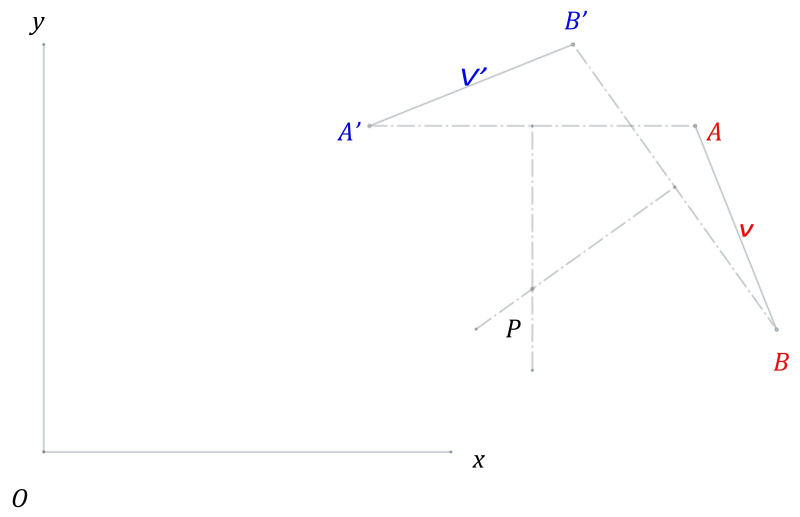
$$P_2 = \begin{pmatrix} R \cdot \left(1 - \cos\frac{\gamma}{R}\right) \\ -R \cdot \sin\frac{\gamma}{R} \end{pmatrix}, \quad \text{with } \frac{\gamma}{R} = \theta$$

   Thus, when moving C to infinity ($R \to \infty$), using the small-angle approximation :

$$\lim_{R\to\infty} P_2 = \lim_{R\to\infty} \begin{pmatrix} R \cdot \left(1 - \cos\frac{\gamma}{R}\right) \\ -R \cdot \sin\frac{\gamma}{R} \end{pmatrix} = \lim_{R\to\infty} \begin{pmatrix} R \cdot (1 - 1) \\ -R \cdot \frac{\gamma}{R} \end{pmatrix} = \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}$$

   Which is equal to the image of P_1 following a translation $\boldsymbol{t} = \binom{t_x}{t_y} = \binom{0}{-\gamma}$. It can be generalized for any point P_1 and any translation $\boldsymbol{t}$.

2. (a) The points A and B, extremities of the vector v, move respectively towards A’ and B’, extremities of the vector v’ ; the center of rotation can be found at the intersection of the perpendicular bisectors of AA’ and BB’:



   (b) The homogeneous matrix of a rotation $\mathbf{R}$ around a point P is :

$$\mathbf{M} = \begin{pmatrix} \mathbf{R}(\theta) & \mathbf{p} - \mathbf{R}(\theta)\mathbf{p} \\ \mathbf{0} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} c & -s & a \\ s & c & b \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore :

$$\mathbf{p} - \mathbf{R}(\theta)\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\Rightarrow \mathbf{p}\big(\mathbf{I} - \mathbf{R}(\theta)\big) = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \big(\mathbf{I} - \mathbf{R}(\theta)\big)^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \right)^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \begin{pmatrix} 1-c & s \\ -s & 1-c \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \frac{1}{det} \begin{pmatrix} 1-c & -s \\ s & 1-c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \frac{1}{2-2c} \begin{pmatrix} a - ac - bs \\ as + b - bc \end{pmatrix}$$

$$\Rightarrow \begin{matrix} p_x = \dfrac{a - ac - bs}{2-2c} \\ p_y = \dfrac{b - bc + as}{2-2c} \end{matrix}$$

(c) The center of pure rotation is the only point that doesn't change during rotation, therefore we have:

$$\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} c & -s & a \\ s & c & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}$$
$$\Rightarrow \begin{cases} p_x = cp_x - sp_y + a \\ p_y = sp_x + cp_y + b \end{cases}$$
$$\Rightarrow \begin{cases} p_x = \dfrac{a - sp_y}{1-c} \\ p_y = \dfrac{b + sp_x}{1-c} \end{cases}$$
$$\Rightarrow \begin{cases} p_x = \dfrac{a - ac - bs}{2-2c} \\ p_y = \dfrac{b - bc + as}{2-2c} \end{cases}$$
$$\Rightarrow \mathbf{p} = \begin{pmatrix} \dfrac{a - ac - bs}{2-2c} \\ \dfrac{b - bc + as}{2-2c} \end{pmatrix}$$

3. Homogeneous matrix of rotation of 60° around the origin:

$$\mathbf{M}_{r,60^\circ} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Homogeneous matrix of translation of 1 in the *x* direction, then rotation of 60° around the origin:

$$\mathbf{M}_{r,60^\circ}\mathbf{M}_t = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 1/2 \\ \sqrt{3}/2 & 1/2 & \sqrt{3}/2 \\ 0 & 0 & 1 \end{pmatrix}$$

5. Homogeneous matrix of rotation of 60° around $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\mathbf{M}_{r,60^\circ,\mathbf{p}} = \begin{pmatrix} \mathbf{R}(60^\circ) & \mathbf{p} - \mathbf{R}(60^\circ)\mathbf{p} \\ \mathbf{0} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{2-1+\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{2-1-\sqrt{3}}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

With :

$$\mathbf{R}(60^\circ) = \begin{pmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

6. We set $\mathbf{M} = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$

Then :

$$\begin{cases} \boldsymbol{v_2} & = \boldsymbol{M v_1} \\ \boldsymbol{w_2} & = \boldsymbol{M w_1} \end{cases}$$

$$\Rightarrow \begin{cases} \begin{pmatrix} \dfrac{1-\sqrt{3}}{2} \\ \dfrac{1-\sqrt{3}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} \dfrac{2-\sqrt{3}}{2} \\ \dfrac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} 2c + 2t_x = 1 - \sqrt{3} \\ 2s + 2t_y = 1 - \sqrt{3} \\ 2c - 2s + 2t_x = 2 - \sqrt{3} \\ 2s + 2c + 2t_y = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c = \dfrac{\sqrt{3}}{2} \\ s = -\dfrac{1}{2} \\ t_x = \dfrac{1}{2} - \sqrt{3} \\ t_y = 1 - \dfrac{\sqrt{3}}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \theta = -30° \\ t_x = \dfrac{1}{2} - \sqrt{3} \\ t_y = 1 - \dfrac{\sqrt{3}}{2} \end{cases}$$

$$\Rightarrow \mathrm{M} = \begin{pmatrix} \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} & \dfrac{1}{2} - \sqrt{3} \\ -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} & 1 - \dfrac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{pmatrix}$$